TRAVELING WAVES OF A NON-LOCAL CONSERVATION LAW

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This dissertation establishes the existence and uniqueness of the traveling waves related to shocks for a non-local scalar conservation law $u_t + (f(u))_x = K * u - u$, where f is an arbitrary continuous differentiable function, $K * u$ stands for the convolution in the spatial variable x, and K is an arbitrary non-negative kernel with unit mass not necessary centered at the origin and with bounded first moment. We first truncate the problem from the real line $\mathbb{R}$ to a finite domain and then add an artificial viscosity so that the problem becomes a second-order elliptic boundary value problem. Utilizing classical techniques, we establish the existence and uniqueness of the boundary value problem. Then we send the boundary points to infinity to extend the result to the whole real line. Finally we send the viscosity to zero and show that the limit is the traveling wave solution to the non-local scalar conservation law problem.

TABLE OF CONTENTS

LIST OF FIGURES

PREFACE

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This work is dedicated to them!

1.0 INTRODUCTION

1.1 BACKGROUND

To describe the motion of a radiating gas in thermo-nonequilibrium in large scales, it was proposed (see [20]) to utilize the Euler system as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ \{\rho(e + \frac{u^2}{2})\}_t + \{\rho u(e + \frac{u^2}{2}) + pu + q\}_x = 0, \\ -q_{xx} + aq + b(\theta^4)_x = 0. \end{cases} \tag{1.1}$$

In (1.1), ρ, u, p, e and θ are, respectively, the mass density, velocity, pressure, internal energy and absolute temperature of the gas, while q represents the radiative heat flux, a and b are given positive constants depending on the gas itself.

The simplest mathematical model in the study of radiating gas is the hyperbolic-elliptic coupled system

$$\begin{cases} u_t + uu_x + q_x = 0, \\ q_{xx} - q = u_x. \end{cases} \qquad 0 \leq t, \quad -\infty < x < \infty \tag{1.2}$$

with u_0 as the initial condition (see [22]). The system of (1.2) is derived as the third-order approximation of the full system (1.1), while the second-approximation gives the viscous Burgers equation $u_t + uu_x = u_{xx}$, and the first-order approximation gives the inviscid

Burgers equation $u_t + uu_x = 0$. Hamer [16] studied these equations in the physical respect, especially for the steady progressive shock-wave solutions[20].

The propagation of small disturbances in a radiating gas has been widely studied. The effects of non-linearity was raised by Hamer [16] (also see [25])from the Euler system. Here the q is given by an integral equation, whereas the Hamer's equation (1.2) is a first order approximation of a perturbation about an equilibrium. The equation $q_{xx} - q = u_x$ can be solved to give

$$q_x = u - K * u, \quad K(z) = \frac{1}{2}e^{-|z|}.$$

Thus, the Hamer's system can be put into the general form of $u_t + (f(u))_x = K * u - u$.

The traveling wave problem was raised by Serre [31]. It is shown by Serre [31] in the case $\int_{\mathbb{R}} zK(z)dz = 0$ that, entropy solutions, if they exist, are unique up to a spatial translation. His proof carries over to the case when $\int_{\mathbb{R}} zK(z)dz \neq 0$.

When $K(z) = \frac{1}{2}e^{-|z|}$, the function $p := K * u$ satisfies $p_{xx} = p - u = K * u - u$, so that the non-local traveling wave problem (1.5) can be transferred to a second order ode equation, for which a phase plane analysis can be used. Schochet and Tadmor [29], and lately Kawashima and Nishibata [18], proved that when $f = \frac{1}{2}u^2$ and $K(z) = \frac{1}{2}e^{-|z|}$, (1.5) admits a solution if and only if $u_- > u_+$, and the solution is continuous if and only if $u_- - u_+ \leqslant \sqrt{2}$. Recently, Lattanzio, Mascia, and Serre [21] extended the result from a single conservation law (1.3) to more general hyperbolic-elliptic systems.

Non-local operators such as a convolution have been used to replace the classical differential operators, such as the Laplacian, in a variety of linear and non-linear models. In fact, many differential equations originate from approximations of non-local models. For example, van der Waals in 1893 [32] first derived a thermodynamical theory involving a non-local operator $K*u-u$, from which, he derived local models using the approximation $K * u - u \approx a\ \Delta u$ where $a = \frac{1}{2}\int_{\mathbb{R}^n} |y|^2 K(|y|)dy$. Non-local models have recently gain popularity, see examples in [1, 4, 5, 6, 8, 11, 12, 13, 24, 33] and the references therein.

When $f(u) = \frac{1}{2}u^2$, (1.3) is a non-local Burger's equation and was studied in [16, 18, 19, 20, 29]; for the particular Hamer equation, e.g., (1.3) with $f(u) = \frac{1}{2}u^2$ and $K(z) = \frac{1}{2}e^{-|z|}$, see [23, 30, 31]. The Burger's equation has a special feature that distinguishes it from any other conservation law with non-quadratic f. In the Burger's equation, one of the physical meaning of u is the velocity of fluid or traffic. Hence, in a moving coordinate system whose origin moves with velocity of s, the corresponding velocity differs from that in the original coordinate system by an additive s. Thus, using a moving frame, one can reduce the general problem to the case $c = 0$ and $u_- = -u_+$, whereas the system of equations are unchanged. In particular, when K is even, the traveling wave profile ϕ is an odd function. Based on this observation Chmaj [10] recently established the existence of a traveling wave.

The method of Chamj [10] is based on a monotonic iteration technique; see the original development in [3]. Note that when $\phi(x) = -\phi(-x)$ and $K(z) = K(-z)$,

$$K * \phi(x) := \int_{\mathbb{R}} K(x-y)\phi(y)dy = \int_0^\infty \Big\{K(|x-y|) - K(x+y)\Big\}\phi(y)\,dy.$$

For the map from $\phi \in C([0,\infty))$ to $K*\phi \in C([0,\infty))$ to be monotonic, it is necessary and sufficient to have $K(|x-y|) \geqslant K(x+y)$ for all $x > 0, y > 0$, i.e. $K(\cdot)$ is non-increasing on $[0,\infty)$. Chmaj discovered that the map from ψ to ϕ, where ϕ solves $\psi\phi_x + \phi = K * \psi$ in $(0,\infty)$ subject to the boundary condition $\phi(\infty) = u_+$, is monotonic. He cleverly introduced a notion of sub/super solution in the sense that $\phi\phi_x + \phi - K*\phi \lesseqgtr 0$ in $(0,\infty)$ for odd ϕ, and demonstrated that $\underline{\phi}(x) = u_+ x/|x|$ is a subsolution and $\bar{\phi}(x) = \frac{2u_+}{\pi}\arctan(\varepsilon x)$ is a supersolution for a sufficiently small positive ε, provided that $K(z) = o(z^{-4})$ as $z \to \infty$. Note that if $(\underline{\phi}, \bar{\phi})$ is a sub/super solution pair, then $\underline{\phi} \leqslant \bar{\phi}$ in $(0,\infty)$ and $\bar{\phi} \leqslant \underline{\phi}$ in $(-\infty, 0)$ since $(\underline{\phi}, \bar{\phi})$ is odd. Hence, such a notion of sub/super solution is different from the classical ones for parabolic equations.

For parabolic differential equations such as $u_t = u_{xx} + f(u)$, the existence, uniqueness, and asymptotic stability of traveling waves has been well-studied; see examples in [2, 15, 27, 28] and the references therein. For non-local problems, Chen developed in [8] a quite general framework for the traveling wave problem of a bistable dynamics $u_t = \mathcal{A}[u]$ where $\mathcal{A}$ is a general non-linear non-local operator, with the fundamental requirement that the system satisfies a comparison principal: If $u_t \geqslant \mathcal{A}[u]$, $v_t \leqslant \mathcal{A}[v]$, and $u(\cdot,0) \geqslant v(\cdot,0)$, then $u(\cdot,t) \geqslant v(\cdot,t)$ for all $t \geqslant 0$.

1.2 PROBLEM STATEMENT AND CONTRIBUTIONS

In this dissertation, we study traveling waves related to shock waves for a class of non-local scalar conservation laws in the form of

$$u_t + (f(u))_x = K * u - u, \qquad x \in \mathbb{R}, t > 0. \tag{1.3}$$

Here f is a C^2 function, $K * \phi(x) := \int_{\mathbb{R}} K(x-y)\phi(y)dy$ stands for the convolution, and K is a non-negative integrable function with unit mass and bounded first moment. More precisely,

$$\begin{aligned} & f \in C^2(\mathbb{R}), \\ & K \geqslant 0 \text{ in } \mathbb{R}, \\ & \textstyle\int_{\mathbb{R}} K(z)\,dz = 1, \\ & \textstyle\int_{\mathbb{R}} |z| K(z)\,dz < \infty. \end{aligned} \tag{1.4}$$

A traveling wave is a solution of the form $u(x,t) = \phi(z)$ where $z = x - ct$ is the coordinate in the moving frame whose origin moves with velocity c. To obtain the traveling wave solution to (1.3), we first perform change of variables to transform (1.3) into an equation about ϕ. Starting from the left-hand side of the equation:

$$\begin{aligned} u_t &= \phi'(z)\frac{\partial z}{\partial t} = \phi'(z)(-c) = -c\phi'(z), \\ \Big(f(u)\Big)_x &= \Big(f(\phi)\Big)'\frac{\partial z}{\partial x} = \Big(f(\phi)\Big)'. \end{aligned}$$

Computation on the right-hand side:

$$(K*u)(x) = \int_{\mathbb{R}} K(x-y)u(y,t)\,dy.$$

Do the substitution $u(y,t) = \phi(l)$ and $l = y - ct$ we have

$$\begin{aligned} (K*u)(x) &= \int_{\mathbb{R}} K(x-y)\phi(l)\,dy = \int_{\mathbb{R}} K(x-l-ct)\phi(l)\,dl \\ &= \int_{\mathbb{R}} K(z-l)\phi(l)\,dl = (K*\phi)(z). \end{aligned}$$

Hence we obtain a new equation about the function $\phi(z)$:

$$-c\phi_z + (f(\phi))_z = K*\phi - \phi,$$

i.e.,

$$(f(\phi) - c\phi)_z + \phi = K*\phi.$$

Of concern are traveling waves that connect the constant states u_- at $z = -\infty$ and u_+ at $z = \infty$. This translates to the following traveling wave problem,

$$\begin{cases} (\,f(\phi) - c\phi\,)_z + \phi = K*\phi \qquad \text{in } \ \mathbb{R}, \\ \phi(-\infty) = u_-\ , \qquad \phi(\infty) = u_+. \end{cases} \tag{1.5}$$

Here, to account for possible discontinuities, the differential equation is understood as

$$f(\phi(z)) - c\phi(z) = C + \int_0^z [K*\phi(y) - \phi(y)]dy, \qquad \forall\, z \in \mathbb{R} \tag{1.6}$$

where $C = f(\phi(0)) - c\phi(0)$ is a constant. Thus, $f(\phi) - c\phi$ is continuous across any discontinuities of ϕ.

Definition 1. *Assume $f, g \in L^1_{loc}(\mathbb{R})$, we say $f = g$ on $\mathbb{R}$ in the distribution sense, if for any non-negative smooth function $\rho \in C^\infty(\mathbb{R})$ with compact support, the following holds:*

$$\int_{\mathbb{R}} f\,\rho\;dx = \int_{\mathbb{R}} g\,\rho\;dx.$$

Similarly, $f < g$ on $\mathbb{R}$ in the distribution sense, if for any non-negative smooth function $\rho \in C^\infty(\mathbb{R})$ with compact support, and $\rho \neq 0$,

$$\int_{\mathbb{R}} f\;\rho\;dx < \int_{\mathbb{R}} g\;\rho\;dx.$$

We calculate, for such ρ from above:

$$\begin{aligned}
\int_{-\infty}^{\infty} \rho\Big(f(\phi) - c\phi\Big)_z\,dz &= -\int_{-\infty}^{\infty} \rho_z\Big(f(\phi) - c\phi\Big)\,dz \\
&= -\int_{-\infty}^{\infty} \rho_z\Big(C + \int_0^z (K * \phi - \phi)\,dy\Big)\,dz \\
&= \int_{-\infty}^{\infty} \rho\Big(\int_0^z (K * \phi - \phi)\,dy\Big)_z\,dz.
\end{aligned}$$

Here, $f(\phi) - c\phi \in C(\mathbb{R})$ means $\Big(f(\phi) - c\phi\Big)_z = K * \phi - \phi$ on $\mathbb{R}$ in the distribution sense.

Consider (1.6) and send z to $\pm\infty$ then take the difference, we find that

$$\frac{f(u_+) - f(u_-)}{u_+ - u_-} - c = \frac{1}{u_+ - u_-}\int_{\mathbb{R}} (K * \phi - \phi)dy$$

In the next section when we talk about the wave speed, we will show that

$$\frac{1}{u_+ - u_-}\int_{\mathbb{R}} (K * \phi - \phi)dy = -\int_{\mathbb{R}} zK(z)\,dz \tag{1.7}$$

Hence, the speed c of the traveling wave is uniquely determined by the extended Rankine-Hugoniot formula

$$c = \frac{f(u_+) - f(u_-)}{u_+ - u_-} + \int_{\mathbb{R}} zK(z)\,dz. \tag{1.8}$$

Definition 2. *Assume* $\int_{\mathbb{R}} K(z)\,dz = 1$*, a traveling wave solution to problem (1.5) is a pair* $(c, \phi) \in \mathbb{R} \times L^\infty(\mathbb{R})$ *such that*

1. $f(\phi(z)) - c\phi(z) = f(\phi(0)) - c\phi(0) + \int_0^z (K * \phi - \phi)(y)\,dy,$
2. $\lim_{z\to-\infty} \phi(z) = u_-, \quad \lim_{z\to\infty} \phi(z) = u_+.$

c is called the wave speed, and ϕ is called the profile. Specifically, when $|zK(z)| \in L^1$*,*

$$c = \frac{f(u_+) - f(u_-)}{u_+ - u_-} + \int_{\mathbb{R}} zK(z)\,dz.$$

From the knowledge of the classical conservation law $u_t + (f(u))_x = 0$, to have a physical solution, it is natural to impose the condition (Figure 1)

$$g(s) := f(s) - f(u_-) + \frac{f(u_+) - f(u_-)}{u_+ - u_-}(u_- - s) < 0, \quad \forall\, s \in (u_+, u_-). \tag{1.9}$$

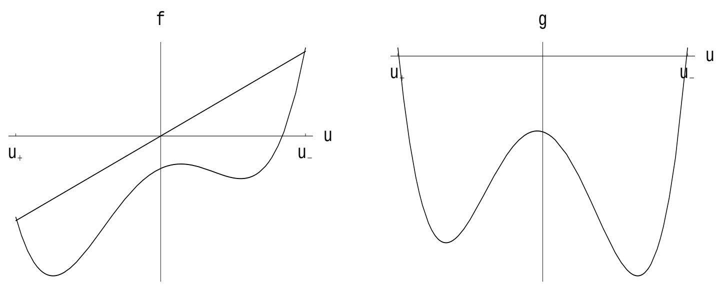

Figure 1: Sketch of f and g.

In this dissertation, the main theorem we shall prove is the following:

Theorem 1. *Assume* (1.4)*. For each pair* $(u_-, u_+) \in \mathbb{R}^2$ *satisfying* $u_- > u_+$ *and* (1.9)*, there exists a solution* (c, ϕ) *to* (1.5) *where* c *is given by* (1.8) *and* $\phi_x \leqslant 0$ *in* $\mathbb{R}$ *in the distribution sense. In addition,* ϕ *is an entropy solution in the sense that for every* $k \in \mathbb{R}$*,*

$$[(F(\phi) - F(k))\mathrm{sgn}(\phi - k)]_x \leq \mathrm{sgn}(\phi - k)(K * \phi - \phi) \tag{1.10}$$

in the distribution sense, where $F(s) := f(s) - cs$*.*

Moreover, the solution has a jump if

$$\max_{s\in[u_+,u_-]} \Big(- g(s) \Big) > (u_- - u_+) \int_{\mathbb{R}} |z|K(z)\,dz. \tag{1.11}$$

Since ϕ is decreasing and $[F(\phi)]_x = K*\phi-\phi$, the entropy condition (1.10) is equivalent to the condition that

$$F(\phi(x+)) = F(\phi(x-)) \geqslant \max_{s\in[\phi(x+),\phi(x-)]} F(s), \qquad \forall\, x \in \mathbb{R}$$

where $\phi(x\pm) = \lim_{y\to x\pm} \phi(y)$.

For the traveling wave problem (1.5), here we develop a technique totally different from that in [10]. We shall first solve a problem, obtained by truncation, in a bounded interval and then take the limit by letting the bounded interval approach $\mathbb{R}$. The problem on the bounded domain is solved in part by Chmaj's idea [10] in his construction of monotonic operator.

For the uniqueness of traveling waves, in general there are infinitely many non-monotonic solutions to (1.5). In certain admissible classes, Serre [31] proved that traveling waves are unique. Here for the completeness of the dissertation, we shall carry out Serre's uniqueness proof for the admissible class

$$\mathcal{A} := \Big\{\phi : \mathbb{R} \to \mathbb{R} \mid \quad \lim_{y\nearrow x} \phi(x) \geqslant \lim_{y\searrow x} \phi(x), \quad \forall\, x \in \mathbb{R}\Big\}.$$

The rest of the dissertation is organized as follows. In Chapter 2, we first show (1.7), then we study the singular perturbed convection-diffusion problem on a finite interval $[-n, n]$ for positive ε and n, imposing the "boundary" condition $\phi = u_-$ in $(-\infty, -n]$ and $\phi = u_+$ in $[n, \infty)$. In Chapter 3, we take the limit as $n \to \infty$ to establish the existence of a traveling wave, and then in Chapter 4, we send $\varepsilon \searrow 0$ to get the existence and finally show that solutions of (1.5) in the class $\mathcal{A}$ are unique (up to a translation).

2.0 A PROBLEM FROM TRUNCATION

2.1 THE WAVE SPEED

We now prove the wave speed satisfies (1.8), following Chen's approach in [7].

Lemma 1. *Assume that*

$$\int_{\mathbb{R}}(1+|z|)|\,K(z)|dz < \infty$$
$$\int_{\mathbb{R}} K(z)dz = 1.$$

Then for every $\psi \in L^{\infty}(\mathbb{R})$ *having limits* $\lim_{x\to\pm\infty}\psi(x) =: \psi(\pm\infty)$, *there holds*

$$\lim_{a\to-\infty, b\to\infty}\int_a^b \Big\{\psi(x) - K*\psi(x)\Big\}dx = \Big[\psi(\infty) - \psi(-\infty)\Big]\int_{\mathbb{R}} zK(z)\,dz.$$

Proof. Given $\int_{\mathbb{R}} K(z)dz = 1$, for $a, b \in \mathbb{R}$, calculate the integral on the left-hand side before taking limits:

$$
\begin{aligned}
\int_a^b \Big\{\psi(x) - K*\psi(x)\Big\}dx &= \int_a^b \Big[\psi(x) - \int_{-\infty}^{\infty} K(x-z)\psi(z)dz\Big]dx \\
&= \int_a^b \Big[\int_{-\infty}^{\infty} K(z)dz\psi(x) - \int_{-\infty}^{\infty} K(x-z)\psi(x)dz\Big]dx \\
&= \int_a^b \Big[\int_{-\infty}^{\infty} K(z)\psi(x)dz - \int_{-\infty}^{\infty} K(z)\psi(x-z)dz\Big]dx \\
&= \int_a^b \Big[\int_{-\infty}^{\infty} K(z)\Big(\psi(x) - \psi(x-z)\Big)dz\Big]dx \\
&= \int_{-\infty}^{\infty} K(z)\Big\{\int_a^b \Big[\psi(x) - \psi(x-z)\Big]dx\Big\}dz \\
&= \int_{-\infty}^{\infty} K(z)\Big[\int_a^b \psi(x)dx - \int_{a-z}^{b-z} \psi(x)dx\Big]dz \\
&= \int_{-\infty}^{\infty} K(z)\Big[\int_{b-z}^{b} \psi(x)dx - \int_{a-z}^{a} \psi(x)dx\Big]dz \\
&= \int_{-\infty}^{\infty} K(z)\Big\{\int_{b-z}^{b} \Big(\psi(x) - u_+\Big)dx - \int_{a-z}^{a} \Big(\psi(x) - u_-\Big)dx - u_- z \\
&\quad +u_+ z\Big\}dz \\
&= (u_+ - u_-)\int_{-\infty}^{\infty} zK(z)dz - \int_{-\infty}^{\infty} K(z)\Big[\int_{a-z}^{a} \Big(\psi(x) - u_-\Big)dx\Big]dz \\
&\quad + \int_{-\infty}^{\infty} K(z)\Big[\int_{b-z}^{b} \Big(\psi(x) - u_+\Big)dx\Big]dz.
\end{aligned}
$$

Denote

$$
\begin{aligned}
I &:= \int_{-\infty}^{\infty} K(z)\Big[\int_{a-z}^{a} \Big(\psi(x) - u_-\Big)dx\Big]dz, \\
II &:= \int_{-\infty}^{\infty} K(z)\Big[\int_{b-z}^{b} \Big(\psi(x) - u_+\Big)dx\Big]dz.
\end{aligned}
$$

Now consider properties of I as $a \to -\infty$.

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} K(z)\Big[\int_{a-z}^{a}\Big(\psi(y)-u_-\Big)dy\Big]dz \\
&= \int_{\mathbb{R}} zK(z)\Big[\int_0^1 \Big(\psi(a-\theta z)-u_-\Big)d\theta\Big]dz \\
&= \int_0^1\int_{\mathbb{R}}\Big[zK(z)\Big(\psi(a-\theta z)-u_-\Big)\Big]dz\ d\theta.
\end{aligned}
$$

For every fixed θ and z, sending a to $-\infty$, $\psi(a-\theta z)\to\psi(-\infty)=u_-$. Since $|xK(x)|\in L^1$, by Lebesgue's Dominated Convergence Theorem [26], $I\to 0$.

Similarly,

$$
\begin{aligned}
II &= \int_{-\infty}^{\infty} K(z)\Big[\int_{b-z}^{b}\Big(\psi(x)-u_+\Big)dx\Big]dz \\
&= \int_{\mathbb{R}} zK(z)\Big[\int_0^1 \Big(\psi(b-\theta z)-u_-\Big)d\theta\Big]dz \\
&= \int_0^1\int_{\mathbb{R}}\Big[zK(z)\Big(\psi(b-\theta z)-u_-\Big)\Big]dz\ d\theta.
\end{aligned}
$$

For every fixed θ and z, sending b to ∞, $\psi(b-\theta z)\to\psi(\infty)=u_+$, $II\to 0$ as well.

Hence,

$$
\int_{-\infty}^{\infty}\Big(K*\psi-\psi\Big)(y)dy=(u_--u_+)\int_{-\infty}^{\infty} zK(z)dz.
$$

This proves the lemma. $\square$

2.2 PROBLEM FROM TRUNCATION

In the dissertation, conditions in (1.4) are always assumed. Also u_-, u_+ are fixed constants with $u_- > u_+$. Denote

$$c := c_f + c_K, \qquad c_f := \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \qquad c_K := \int_{\mathbb{R}} zK(z)\,dz,$$
$$g(s) := f(s) - f(u_-) - c_f[s - u_-].$$

Note that

$$g(u_+) = g(u_-) = 0.$$

$$g(s) < 0, \quad \forall s \in (u_+, u_-)$$

as shown in the Figure 1.

For positive ε and n, we consider the "boundary" value problem

$$\begin{cases} -\varepsilon u_{xx} + [f(u) - cu]_x + u = K * u & \text{in } \ (-n, n), \\ u = u_- \ \text{ in } \ (-\infty, -n], \qquad u = u_+ \text{ in } \ [n, \infty). \end{cases} \tag{2.1}$$

Theorem 2. *Assume (1.4) and $u_+ < u_-$. Then for every positive ε and n, (2.1) admits a unique solution $u \in C(\mathbb{R}) \cap C^2([-n-, n+])$. In addition, the solution satisfies $u_x < 0$ on $[-n-, n+]$ and for all $x \in [-n, n]$,*

$$-\varepsilon u_x(x) + g(u(x)) + \int_{\mathbb{R}} K(z) \int_{x-z}^{x} [u(y) - u(x)]\,dy\,dz = C(\varepsilon, n) \tag{2.2}$$

where $C(\varepsilon, n)$ is a constant satisfying

$$0 < C(\varepsilon, n) \leqslant (u_- - u_+)\Big\{\frac{\varepsilon}{2n} + \int_{\mathbb{R}} |z|K(z)\,dz\Big\} + \max_{s\in[u_+,u_-]} g(s).$$

Proof. We divide the proof into several steps in the following sections.

2.3 EXISTENCE

We shall employ the Schauder's fixed point theorem [14]. For this we use the Banach space $\mathbf{X} = C([-n,n])$ and the set

$$D := \{v \in \mathbf{X} \mid u_+ \leqslant v(x) \leqslant u_-, \quad \forall x \in [-n,n]\}.$$

Claim 1. *D is a closed, bounded, and convex subset of $\mathbf{X}$.*

Proof. We divide the proof into three parts.

1. D is closed if whenever $\left\{w_n\right\}_{n=1}^{\infty}$ is a sequence of elements of D and w_n uniformly converges to w, then w is also an element of D. In fact, assume such a sequence, $\forall w_n \in D \subset X$ which is a Banach Space, so $w \in X$. $\forall w_n \in D, \forall x \in [-n,n], u_+ \leq w_n(x) \leq u_-$, so $u_+ \leq w(x) \leq u_-$. Because $w \in X$, $w \in D$, which implies that D is closed.

2. D is bounded if there exists real value M such that $||w|| < M$ for all element w found in D. For our case here, $M := max(|u_-|, |u_+|)$ and this sequence is totally bounded.

3. D is convex if $\forall\, t \in [0,1], \forall\, w, v \in D, t\, w + (1-t)\, v \in D$. Indeed, $w, v \in D$ implies that $w, v \in X$, and $t\, w + (1-t)\, v \in X$ since X is a Banach Space. Because $u_+ \leq w, v \leq u_-$, one can easily calculate that $u_+ \leq t\, w + (1-t)\, v \leq u_-$. Hence $t\, w + (1-t)\, v \in D$ and D is convex.

   This completes the proof of Claim 1. □

For every $v \in D$, we extend the definition domain of v from $[-n,n]$ to $\mathbb{R}$ by setting

$$v = u_- \quad \text{in} \quad (-\infty, -n), \qquad v = u_+ \quad \text{in} \quad (n, \infty). \tag{2.3}$$

Claim 2. *$K * v(x) = \int_{\mathbb{R}} K(x-y)v(y)\,dy$ is well-defined and continuous on $\mathbb{R}$ provided K is a non-negative integrable function and $\int_{\mathbb{R}} K(z)dz = 1, |z|K(z) \in L^\infty(\mathbb{R})$, and $v \in D$ with extended definition domain as in (2.3).*

If f and g are compactly supported continuous functions, then their convolution exists, and is also compactly supported and continuous [17]. It follows that if either function (K in our case) is compactly supported and the other is locally integrable (v in our case), then the convolution ($K * v$) is well-defined and continuous.

Under the previous assumptions, we can prove that the boundary value problem of the second-order linear ordinary differential equation (ODE)

$$\begin{cases} -\varepsilon u_{xx} + [f'(v) - c]u_x + u = K * v \qquad \text{in} \;\; (-n, n), \\ u(-n) = u_-, \qquad u(n) = u_+ \end{cases} \tag{2.4}$$

have a unique solution.

In fact, (2.4) is an second order (elliptic) differential equation with two boundary conditions. We denote the linear operator $\mathcal{L} : u \to \mathcal{L}u = -\varepsilon u'' + p\,u' + u$ as

$$\mathcal{L} = -\varepsilon \frac{d^2}{dx^2} + p\frac{d}{dx} + 1,$$

where $p := f'(v) - c$. Since $f \in C^2(\mathbb{R})$ and c is a constant, p is a continuous function. Note the other important fact is that ε is a positive constant. For this kind of second-order differential equations, we have the following important properties, while the proof can be found in [9].

Theorem 3. *(Maximum Principle). Let* $\Omega = (a, b)$, $\overline{\Omega} = [a, b]$, *and* $\partial\Omega = \{a, b\}$. *If* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfies*

$$\begin{cases} \mathcal{L}u \geqslant 0 & \text{in } \Omega, \\ u \geqslant 0 & \text{on } \partial\Omega, \end{cases}$$

then either $u \equiv 0$ *in* Ω *or* $u > 0$ *in* Ω.

Theorem 4. *(Comparison Principle). Let* $\Omega = (a, b)$ *and assume that* $p \in C(\overline{\Omega}; \mathbb{R})$ *and* $\varepsilon > 0$ *in* Ω. *Suppose* $\mathcal{L}u \geqslant \mathcal{L}v$ *in* Ω *and* $u \geqslant v$ *on* $\partial\Omega$. *Then either* $u \equiv v$ *or* $u > v$ *in* Ω.

Now we are to prove the following theorem.

Theorem 5. *Assume* $\varepsilon > 0$, f, K *as in (1.4),* $v \in D$ *with extended definition domain as in (2.3), there exists a unique solution* $u \in C^2(-n, n)$ *to the boundary value problem (2.4).*

Proof. We first prove the existence. Because it is known that every second-order ordinary differential equation with initial values has a unique solution, let's assume ϕ_1 and ϕ_2 be, respectively, the unique solution to the homogeneous problem

$$-\varepsilon u_{xx} + [f'(v) - c]u_x + u = 0 \qquad \text{in} \;\; (-n, n),$$

subject to the initial values:

$$\phi_1(-n) = 0, \quad \phi_1'(-n) = -1, \quad \phi_2(n) = 0, \quad \phi_2'(n) = -1.$$

By the maximum principle, the non-trivial solution ϕ_1 can only obtain non-negative maximum or non-positive minimum at the boundary, $\phi_1(x) < 0 \; \forall x \in (-n, n]$. Hence $\phi_1(n) \neq \alpha\phi_2(n)$ for any α, and it follows that ϕ_1 and ϕ_2 are two linearly independent solutions, i.e.,

$$\phi_1'(y)\phi_2(y) - \phi_1(y)\phi_2'(y) < 0, \qquad \text{for all } y \in [-n, n].$$

Follow the idea of Green's function, we claim the solution is in the form of

$$\begin{aligned} u(x) &= (u_-)\frac{\phi_2(x)}{\phi_2(-n)} + (u_+)\frac{\phi_1(x)}{\phi_1(n)} + \int_{-n}^{x} \frac{k*v(y)\phi_1(y)\phi_2(x)}{\phi_1'(y)\phi_2(y)-\phi_1(y)\phi_2'(y)}\,dy \\ &\quad + \int_{x}^{n} \frac{k*v(y)\phi_2(y)\phi_1(x)}{\phi_1'(y)\phi_2(y)-\phi_1(y)\phi_2'(y)}\,dy. \end{aligned} \tag{2.5}$$

Here we provide the verification.

First of all, $u(-n) = u_-,\ u(n) = u_+$, the boundary conditions are satisfied.

$$\begin{aligned} u'(x) &= (u_-)\frac{\phi_2'(x)}{\phi_2(-n)} + (u_+)\frac{\phi_1'(x)}{\phi_1(n)} + \int_{-n}^{x} \frac{k*v(y)\phi_1(y)\phi_2'(x)}{\phi_1'(y)\phi_2(y)-\phi_1(y)\phi_2'(y)}\,dy \\ &\quad + \int_{x}^{n} \frac{k*v(y)\phi_2(y)\phi_1'(x)}{\phi_1'(y)\phi_2(y)-\phi_1(y)\phi_2'(y)}\,dy. \end{aligned}$$

$$\begin{aligned} u''(x) &= (u_-)\frac{\phi_2''(x)}{\phi_2(-n)} + (u_+)\frac{\phi_1''(x)}{\phi_1(n)} + \int_{-n}^{x} \frac{k*v(y)\phi_1(y)\phi_2''(x)}{\phi_1'(y)\phi_2(y)-\phi_1(y)\phi_2'(y)}\,dy \\ &\quad + \int_{x}^{n} \frac{k*v(y)\phi_2(y)\phi_1''(x)}{\phi_1'(y)\phi_2(y)-\phi_1(y)\phi_2'(y)}\,dy - k*v(x). \end{aligned}$$

As we assumed

$$\varepsilon\phi_i'' + (f'(v)-c)\phi_i' + \phi_i = 0, \quad i = 1,2,$$

direct calculation gives

$$\varepsilon u'' + (f'(v)-c)u' + u = k*v,$$

i.e., (2.5) gives a solution to (2.4).

Then we prove that there exists at most one solution. Suppose u_1 and u_2 are two solutions to (2.4). Denote $\phi = u_1 - u_2$. Then ϕ is a solution to

$$\begin{cases} -\varepsilon u_{xx} + [f'(v)-c]u_x + u = 0 & \text{in } (-n,n), \\ u(-n) = 0, \qquad u(n) = 0. \end{cases}$$

Apply the maximum principle, since the non-negative maximum and non-positive minimum are obtained on the boundary, we conclude that $\phi \equiv 0$, i.e. $u_1 \equiv u_2$. This completes the proof. $\square$

We define $\mathbf{T}[v] := u$ as the unique solution to (2.4).

Claim 3. $\mathbf{T}[D] \subset D$.

Proof. Since $\displaystyle\int_{\mathbb{R}} K(z)\,dz = 1$ and $u_+ \leq v \leq u_-$, for all $x \in \mathbb{R}$, we have

$$u_+ = \int_{\mathbb{R}} u_+ \, K(z)\,dz \leqslant \int_{\mathbb{R}} v(x-z)K(z)\,dz \leqslant \int_{\mathbb{R}} u_- K(z)\,dz = u_-$$

i.e., $u_+ \leqslant K * v \leqslant u_-$ on $\mathbb{R}$.
If we denote $\mathbf{T}[u_+]$ be the solution to

$$-\varepsilon u_{xx} + [f'(u_+) - c]u_x + u = K * u_+ = u_+ \qquad \text{in} \;\; (-n, n),$$

and denote $\mathbf{T}[u_-]$ in the similar way. Then

$$u_+ = \mathbf{T}[u_+] \leq \mathbf{T}[v] \leq \mathbf{T}[u_-] = u_- \ .$$

The comparison shows that $u = \mathbf{T}[v]$ satisfies $u_+ \leqslant u \leqslant u_-$ in $[-n, n]$, and $u \in X$, so $u \in D$. This gives $\mathbf{T}[D] \subset D$. $\square$

Also, an elliptic estimate shows that $\{\mathbf{T}[v]\}_{v \in D}$ is a bounded set in $C^2([-n, n])$ so $\mathbf{T}$ is compact. Since $f \in C^2$, it is easy to see that $\mathbf{T}$ is continuous. Hence, by the Schauder fixed point theorem, $\mathbf{T}$ admits a fixed point in D, which gives a solution to (2.1).

2.4 AN L^∞ ESTIMATE

Let u be an arbitrary solution to (2.1). Set $M = \max_{x\in[-n,n]} u(x)$ and $w = M - u$. Then $w \geqslant 0$ in $[-n,n]$ and

$$-\varepsilon w_{xx} + [f'(w) - c]w_x + w = K * w \geqslant 0 \text{ in } (-n,n),$$

so by the maximum principle (Theorem 3), $w > 0$ in $(-n,n)$. This implies that

$$M = u_- \quad \text{and} \quad u < u_- \quad \text{in } \ (-n,n).$$

Similarly, $u > u_+$ in $(-n,n)$. Hence,

$$u_+ < u(x) < u_-, \qquad \forall\, x \in (-n,n).$$

2.5 UNIQUENESS

We shall use the following weak version of the Harnack inequality:

Theorem 6. *Let $\Omega = (a,b)$ and $\bar\Omega = [a,b]$. Assume that $p, q \in C(\bar\Omega), w \in C(\bar\Omega)\cap C^2(\Omega)$, $w \geqslant 0$ on $\bar\Omega$ and $-\varepsilon w_{xx} + pw_x + qw \geqslant 0$ in Ω. Then either $w \equiv 0$ or $w > 0$ in Ω.*

Let $u = u_1$ and $u = u_2$ be two arbitrary solutions to (2.1). Define

$$\xi := \inf\{t \geqslant 0 \mid u_1(\cdot - h) \geqslant u_2(\cdot), \quad \forall\, h \geqslant t\}.$$

This constant is well-defined and $\xi \in [0, 2n]$, since for $t = 2n$ and every $h \geqslant t$, $u_1(\cdot - h) \geqslant u_2(\cdot)$. Indeed, we have

$$u_1(x-h) - u_2(x) = u_- - u_2(x) \geqslant 0, \quad \forall\, x \leqslant -n,$$

and

$$u_1(x-h) - u_2(x) = u_1(x-h) - u_+ \geqslant 0, \quad \forall\, x \geqslant n.$$

Claim 4. $\xi = 0$.

Proof. Suppose on the contrary that $\xi > 0$. Consider the function

$$w(x) := u_1(x-\xi) - u_2(x), \qquad x \in \bar{I} := [-n+\xi, n].$$

By continuity, we have $w \geqslant 0$ on $\bar{I}$. Also, using the L^∞ estimate we see that

$$w(n) = u_1(n-\xi) - u_+ > 0, \qquad w(-n+\xi) = u_- - u_2(-n+\xi) > 0.$$

Furthermore, $\forall\, x \in I = (-n+\xi, n)$, since

$$\begin{aligned}
&-\varepsilon u_1''(x-\xi) + (f'(u_1(x-\xi)) - c)u_1'(x-\xi) + u_1(x-\xi) = K * u_1(x-\xi),\\
&-\varepsilon u_2''(x) + (f'(u_2(x)) - c)u_2'(x) + u_2(x) = K * u_2(x).
\end{aligned}$$

Take the difference term by term:

$$\begin{aligned}
\varepsilon w''(x) &= -\varepsilon(u_1''(x-\xi) - u_2''(x)),\\
K * w(x) &= K * (u_1(x-\xi) - u_2(x)) = K * u_1(x-\xi) - K * u_2(x).
\end{aligned}$$

And the first order terms need more calculations:

$$\begin{aligned}
& (f'(u_1(x-\xi)) - c)u_1'(x-\xi) - (f'(u_2(x)) - c)u_2'(x)\\
= \; & f'(u_1(x-\xi))\; u_1'(x-\xi) - c\; u_1'(x-\xi) - f'(u_2(x))\; u_2'(x) + c\; u_2'(x)\\
= \; & f'(u_1(x-\xi))\; u_1'(x-\xi) - f'(u_2(x))\; u_1'(x-\xi) + f'(u_2(x))(u_1'(x-\xi) - u_2'(x))\\
& -c(u_1'(x-\xi) - u_2'(x))\\
= \; & \Big[f'(u_2(x) + \; w(x)) - f'(u_2(x))\Big] u_1'(x-\xi) + \Big[f'(u_2(x)) - c\Big] w'(x)\\
= \; & \left(\int_0^1 f''(u_2(x) + t\; w(x))dt\right) w(x) + \Big[f'(u_2(x)) - c\Big] w'(x).
\end{aligned}$$

Denote $p(x) = [f'(u_2(x)) - c]$ and $q(x) = 1 + \int_0^1 f''(u_2(x) + tw(x))dt$, we obtain

$$-\varepsilon w_{xx} + pw_x + qw = K * w \geqslant 0 \quad \text{in } I = (-n + \xi, n)$$

As $w \geqslant 0$, by the weak Harnack inequality, $w > 0$ on $[-n + \xi, n]$. Consequently, by continuity, there exists $\delta > 0$ such that

$$\min_{\tilde{\delta} \in [0,\delta]} u_1(x - \xi + \tilde{\delta}) > u_2(x) \quad \forall\, x \in [-n + \xi - \delta, n].$$

As $u_1 = u_-$ on $(-\infty, -n]$ and $u_2 = u_+$ on $[n, \infty)$, it implies that

$$u_1(\cdot - h) - u_2(\cdot) \geqslant 0, \quad \forall h \in [\xi - \delta, \xi].$$

Hence it is valid for all $h \in [\xi - \delta, \infty)$ for some $\delta > 0$. But this contradicts the definition of ξ, which means that $\xi = 0$. $\square$

From above, we conclude that $\xi = 0$ and $u_1(\cdot - h) \geqslant u_2(\cdot)$ on $\mathbb{R}$ for all $h \geqslant 0$. Exchanging the roles of u_1 and u_2 we conclude that $u_1 \equiv u_2$. Thus, (2.1) admits a unique solution.

2.6 MONOTONICITY

Let u be the unique solution to (2.1). The previous step concludes that

$$u(\cdot - h) \geqslant u(\cdot), \quad \forall h \geqslant 0,$$

so that

$$u_x \leqslant 0 \quad \text{ in } [-n^+, n^-].$$

By differentiation, one obtains that the function $\zeta := u_x$ satisfies

$$-\varepsilon\zeta_{xx} + [f'(u) - c]\zeta_x + [1 + f''(u)\zeta]\zeta = K * \zeta \leqslant 0.$$

The Harnack inequality and the Hopf's Lemma then imply that

$$u_x = \zeta < 0 \quad \text{ in } [-n^+, n^-].$$

2.7 THE INTEGRAL IDENTITY

Note that

$$u(x) - K * u(x) = \int_{\mathbb{R}} K(z)[u(x) - u(x-z)]dz = \frac{d}{dx}\int_{\mathbb{R}} K(z)\int_{x-z}^{x} u(y)\,dy\,dz.$$

Thus, integrate the equation

$$-\varepsilon u_{xx} + [f(u) - cu]_x + u - K * u = 0,$$

we get

$$-\varepsilon u_x + f(u) - cu + \int_{\mathbb{R}} K(z)\int_{x-z}^{x} u(y)\,dy\,dz = C_1(\varepsilon, n)$$

for some constant $C_1(\varepsilon, n)$. Under the assumption that

$$
\begin{aligned}
&g(u) = f(u) - f(u_-) - c_f[u - u_-],\\
&c = c_f + c_K \quad \text{ where } c_f = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad c_K = \int_{\mathbb{R}} zK(z)\,dz,
\end{aligned}
$$

we can transform it to

$$
-\varepsilon u_x + g(u) - \int_{\mathbb{R}} zK(z)\,dz \cdot u(x) + \int_{\mathbb{R}} K(z) \int_{x-z}^{x} u(y)\,dy\,dz = C(\varepsilon, n)
$$

for some constant $C(\varepsilon, n)$. We therefore obtain

$$
-\varepsilon u_x + g(u) + \int_{\mathbb{R}} K(z) \int_{x-z}^{x} [u(y) - u(x)] = C(\varepsilon, n), \quad \forall\, x \in [-n, n],
$$

which is (2.2).

Now we estimate the size of $C(\varepsilon, n)$. First evaluating (2.2) at $x = n$ and using $g(u_+) = 0$, $u(y) = u_+$ for $y \leq n$, we obtain

$$
\begin{aligned}
C(\varepsilon, n) &= -\varepsilon u_x(n) + \int_{\mathbb{R}} K(z) \int_{n-z}^{n} [u(y) - u_+]dydz\\
&= -\varepsilon u_x(n) + \int_{0}^{\infty} K(z) \int_{n-z}^{n} [u(y) - u_+]\,dy\,dz.
\end{aligned}
$$

Because $u_x(n) < 0$, $K(z) \leq 0$, $u(y) \leq 0$ and $C(\varepsilon, n) > 0$.

Next, we use the mean value theorem to conclude that there exists $\hat{x} \in (-n, n)$ such that

$$
u_x(\hat{x}) = \frac{u(n) - u(-n)}{2n} = \frac{u_+ - u_-}{2n}.
$$

It implies that

$$
\begin{aligned}
C(\varepsilon, n) &= -\varepsilon u_x(\hat{x}) + g(u(\hat{x})) + \int_{\mathbb{R}} K(z) \int_{\hat{x}-z}^{\hat{x}} [u(y) - u(\hat{x})]\,dy\,dz\\
&\leqslant (u_- - u_+)\Big\{\frac{\varepsilon}{2n} + \int_{\mathbb{R}} |z|K(z)\,dz\Big\} + \max_{s \in [u_+, u_-]} g(s).
\end{aligned}
$$

This completes the proof of Theorem 2. $\square$

3.0 THE LIMIT AS $N \to \infty$

In this chapter, we assume, in addition to (1.4), the condition (1.9). We consider the limit as $n \to \infty$ of solution to (2.1) to establish the following:

Theorem 7. *Assume* (1.4), $u_+ < u_-$, *and* (1.9). *Then for every* $\varepsilon > 0$, *there exists a uniuque solution* $u^\varepsilon \in C^2(\mathbb{R})$ *such that*

$$\begin{cases} -\varepsilon u^\varepsilon_{xx} + [f(u^\varepsilon) - cu^\varepsilon]_x + u^\varepsilon = K * u^\varepsilon \quad \textit{in} \;\; \mathbb{R}, \\ u^\varepsilon(0) = \frac{1}{2}(u_+ + u_-), \qquad \lim\limits_{x \to \pm\infty} u^\varepsilon(x) = u_\pm. \end{cases} \tag{3.1}$$

In addition, the solution satisfies $u^\varepsilon_x < 0$ *in* $\mathbb{R}$ *and*

$$-\varepsilon u^\varepsilon_x + g(u^\varepsilon(x)) + \int_{\mathbb{R}} K(z) \int_{x-z}^{x} [u^\varepsilon(y) - u^\varepsilon(x)]\, dy\, dz = 0, \quad \forall\, x \in \mathbb{R}. \tag{3.2}$$

Proof. Again, we divide the proof into two steps.

3.1 EXISTENCE

For each positive n, we denote by $u^{\varepsilon,n}$ the unique solution to (2.1). As $u^{\varepsilon,n}_x < 0$ in $(-n,n)$, there exists a unique $z^{\varepsilon,n} \in (-n,n)$ such that

$$u^{\varepsilon,n}(z^{\varepsilon,n}) = \tfrac{1}{2}(u_+ + u_-).$$

Set

$$v^{\varepsilon,n}(x) := u^{\varepsilon,n}(x + z^{\varepsilon,n}), \qquad \forall\, x \in \mathbb{R},$$
$$b^{\varepsilon,n} = n - z^{\varepsilon,n}, \qquad a^{\varepsilon,n} = -n - z^{\varepsilon,n}.$$

Then

$$v^{\varepsilon,n}(0) = \tfrac{1}{2}(u_- + u_+),$$
$$v^{\varepsilon,n} = u_+ \text{ in } [b^{\varepsilon,n}, \infty),$$
$$v^{\varepsilon,n} = u_- \text{ in } (-\infty, a^{\varepsilon,n}]$$

and

$$-\varepsilon v^{\varepsilon,n}_x(x) + g(v^{\varepsilon,n}(x)) + \int_{\mathbb{R}} K(z) \int_{x-z}^{x} [v^{\varepsilon,n}(y) - v^{\varepsilon,n}(x)]\, dy\, dz = C(\varepsilon,n), \quad \forall\, x \in [a^{\varepsilon,n}, b^{\varepsilon,n}].$$

Now consider the family $\{v^{\varepsilon,n}\}_{n\geqslant 1}$. This is an equicontinuous family since $u^{\varepsilon,n}$ has bounded derivative. Hence, there exist a subsequence $\{n_j\}_{j=1}^{\infty}$, constants $a^{\varepsilon}, b^{\varepsilon}, c^{\varepsilon}$, and a function $u^{\varepsilon} \in C(\mathbb{R}) \cap C^2((a^{\varepsilon}, b^{\varepsilon}))$ such that

$$\begin{aligned}
\lim_{j\to\infty} n_j &= \infty, \\
\lim_{j\to\infty} b^{\varepsilon,n_j} &= b^{\varepsilon} \in (0,\infty], \\
\lim_{j\to\infty} a^{\varepsilon,n_j} &= a^{\varepsilon} \in [-\infty, 0), \qquad \Big(|a^{\varepsilon}| + b^{\varepsilon} = \infty\Big), \\
\lim_{j\to\infty} C(\varepsilon, n_j) &= c^{\varepsilon}, \qquad \Big(0 \leqslant c^{\varepsilon} \leqslant [u_- - u_+] \int_{\mathbb{R}} |z| K(z)\, dz, \text{ since } \max_{s\in[u_+,u_-]} g(s) \leq 0\Big), \\
\lim_{j\to\infty} v^{\varepsilon,n_j} &= u^{\varepsilon} \quad \text{in } \ C([-M,M]) \cap C^2((a,b)), \ \forall\, M > 0, a^{\varepsilon} < a < b < b^{\varepsilon}.
\end{aligned}$$

Hence, from the integral differential equation for $v^{\varepsilon,n}$ we derive that

$$- \varepsilon u^\varepsilon_x(x) + g(u^\varepsilon(x)) + \int_{\mathbb{R}} K(z) \int_{x-z}^{x} [u^\varepsilon(y) - u^\varepsilon(x)]\, dy\, dz = c^\varepsilon, \quad \forall\, x \in (a^\varepsilon, b^\varepsilon). \tag{3.3}$$

After differentiation, we see that $u^\varepsilon \in C(\mathbb{R}) \cap C^3((a^\varepsilon, b^\varepsilon))$ and

$$-\varepsilon u^\varepsilon_{xx} + [f(u^\varepsilon) - cu^\varepsilon]_x + u^\varepsilon = K * u^\varepsilon \qquad \text{in } \ (a^\varepsilon, b^\varepsilon).$$

Now to show that u^ε satisfies (3.1) and (3.2), it suffices to show that

$$b^\varepsilon = \infty, \quad a^\varepsilon = -\infty, \quad c^\varepsilon = 0, \qquad u^\varepsilon(\pm\infty) = u_\pm. \tag{3.4}$$

Since $|a^\varepsilon| + b^\varepsilon = \infty$, either $b^\varepsilon = \infty$ or $a^\varepsilon = -\infty$.

First we consider the case that $b^\varepsilon = \infty$. Since $u^\varepsilon_x \leqslant 0$ in $\mathbb{R}$ and u^ε bounded below, $\lim\limits_{x\to\infty} u^\varepsilon(x)$ exists. Take a sequence $\{x_j\}$ such that $x \to \infty$ as $j \to \infty$. By the mean value theorem, $u^\varepsilon_x(x_j) \to \infty$ as $j \to \infty$. Hence we obtain from (3.3) that

$$g(u^\varepsilon(\infty)) = c^\varepsilon.$$

As $c^\varepsilon \geqslant 0$ and $g(s) < 0$ for all $s \in (u_+, u_-)$, we conclude that

$$c^\varepsilon = 0, \qquad u^\varepsilon(\infty) = u_+.$$

Now should $|a^\varepsilon| < \infty$, we would have $u^\varepsilon = u_-$ on $(-\infty, a^\varepsilon]$. Using the differential equation $-\varepsilon u^\varepsilon_{xx} + [f'(u^\varepsilon) - c]u^\varepsilon_x + u^\varepsilon = K * u^\varepsilon$ in $[a^{\varepsilon-}, \infty)$ and Hopf's Lemma we also conclude that $u_x(a^{\varepsilon-}) < 0$. From which, we obtain

$$\begin{aligned} 0 = c^\varepsilon &= -\varepsilon u^\varepsilon_x(a^{\varepsilon-}) + g(u_-) + \int_{\mathbb{R}} K(x-z) \int_{a^\varepsilon - z}^{a^\varepsilon} [u(y^\varepsilon) - u_-]\, dy\, dz \\ &= -\varepsilon u^\varepsilon_x(a^{\varepsilon-}) + \int_{-\infty}^{0} K(z) \int_{a^\varepsilon}^{a^\varepsilon - z} [u_- - u^\varepsilon(y)]\, dy\, dz > 0, \end{aligned}$$

which is a contradiction. Thus, $a^\varepsilon = -\infty$.

Then by sending $x \to -\infty$ we conclude from (3.3) that $g(u(-\infty)) = 0$, so that $u^\varepsilon(-\infty) = u_-$. Thus, (3.4) holds when $b^\varepsilon = \infty$.

In a similarly manner, we can show that (3.4) holds when $a^\varepsilon = -\infty$. Hence, we obtain a solution to (3.1). The solution satisfies (3.2).

Finally, set $\zeta = u^\varepsilon_x$. Then $\zeta \leqslant 0$ in $\mathbb{R}$ and, from (3.1),

$$-\varepsilon\zeta_{xx} + [f'(u) - c]\zeta_x + [f''(u^\varepsilon)\zeta + 1]\zeta = K * \zeta \leqslant 0.$$

The weak Harnack inequality then implies that $u^\varepsilon_x = \zeta < 0$ in $\mathbb{R}$.

3.2 UNIQUENESS

Let u^ε and v^ε be two arbitrary solutions to (3.1). Set $\zeta = v^\varepsilon - u^\varepsilon$. Then

$$\begin{aligned}
K * \zeta &= K * (v^\varepsilon - u^\varepsilon) = -\varepsilon(v^\varepsilon - u^\varepsilon)_{xx} + (f(v^\varepsilon) - f(u^\varepsilon) - c(v^\varepsilon - u^\varepsilon))_x + v^\varepsilon - u^\varepsilon \\
&= -\varepsilon\zeta_{xx} + \left(\int_0^1 f'(u^\varepsilon(x) + t\zeta(x))\,dt \cdot \zeta - c\zeta\right)_x + \zeta \\
&= -\varepsilon\zeta_{xx} + \left[\left(\int_0^1 f'\Big(u^\varepsilon(x) + t\zeta(x)\Big)dt - c\right)\zeta\right]_x + \zeta,
\end{aligned}$$

i.e.,

$$-\varepsilon\zeta_{xx} + [G\zeta]_x + \zeta - K * \zeta = 0 \quad \text{in } \mathbb{R}, \tag{3.5}$$

where $G(x) = \displaystyle\int_0^1 f'(u^\varepsilon(x) + t\zeta(x))dt - c$.

Let $\rho \in C_c^\infty(\mathbb{R})$ be a non-negative smooth function with compact support in $\mathbb{R}$. Let $\Phi \in C^\infty(\mathbb{R})$ be an arbitrary function. By multiplying (3.5) by $\Phi(\zeta(x))\rho(x)$ and integrating the resulting equation over $\mathbb{R}$ we obtain

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} \rho\Phi(\zeta)\Big\{-\varepsilon\zeta_{xx} + [G\zeta]_x + \zeta - K*\zeta\Big\}dx \\
&= \Big(-\varepsilon\Phi(\zeta)\zeta_x\rho + \Phi(\zeta)G\zeta\rho\Big)\Big|_{-\infty}^{\infty} + \int_{\mathbb{R}} \rho\Big\{\varepsilon\Phi'(\zeta)\zeta_x^2 - \Phi'(\zeta)G\zeta\zeta_x\Big\}dx \\
&\quad + \int_{\mathbb{R}} [\varepsilon\Phi(\zeta)\zeta_x - \Phi(\zeta)G\zeta]\rho_x\,dx + \int_{\mathbb{R}} \rho\Big\{\Phi(\zeta)\zeta - \Phi(\zeta)K*\zeta\Big\}dx \\
&= \int_{\mathbb{R}} \rho\Big\{\varepsilon\Phi'(\zeta)\zeta_x^2 - \Phi'(\zeta)G\zeta\zeta_x\Big\}dx + \int_{\mathbb{R}} [\varepsilon\Phi(\zeta)\zeta_x - \Phi(\zeta)G\zeta]\rho_x\,dx \\
&\quad + \int_{\mathbb{R}} \rho\Big\{\Phi(\zeta)\zeta - \Phi(\zeta)K*\zeta\Big\}dx \\
&= \int_{\mathbb{R}} \rho\Big\{\varepsilon\Phi'(\zeta)\zeta_x^2 - \Phi'(\zeta)G\zeta\zeta_x + \Phi(\zeta)\zeta - \Phi(\zeta)K*\zeta\Big\}dx + \int_{\mathbb{R}} [\varepsilon\Phi(\zeta)\zeta_x - \Phi(\zeta)G\zeta]\rho_x\,dx.
\end{aligned}
$$

Now we take

$$\Phi(s) := \Phi_\delta(s) := \frac{s}{\sqrt{s^2+\delta^2}}, \quad \delta > 0.$$

Then

$$\Phi_\delta'(s) = \frac{\delta^2}{(s^2+\delta^2)^{3/2}},$$

so that

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} \rho\Big\{\varepsilon\Phi_\delta'(\zeta)\zeta_x^2 - \Phi_\delta'(\zeta)G\zeta\zeta_x + \Phi_\delta(\zeta)\zeta - \Phi_\delta(\zeta)K*\zeta\Big\}dx + \int_{\mathbb{R}} [\varepsilon\Phi_\delta(\zeta)\zeta_x - \Phi_\delta(\zeta)G\zeta]\rho_x\,dx \\
&\geqslant \int_{\mathbb{R}} \rho\Big\{-\Phi_\delta'(\zeta)G\zeta\zeta_x + \Phi_\delta(\zeta)\zeta - \Phi_\delta(\zeta)K*\zeta\Big\}dx + \int_{\mathbb{R}} [\varepsilon\Phi_\delta(\zeta)\zeta_x - \Phi_\delta(\zeta)G\zeta]\rho_x\,dx \\
&= \int_{\mathbb{R}} \rho\Big\{-\frac{\delta^2 G\zeta\zeta_x}{(\zeta^2+\delta^2)^{3/2}} + \frac{\zeta^2}{\sqrt{\zeta^2+\delta^2}} - \frac{\zeta\cdot K*\zeta}{\sqrt{\zeta^2+\delta^2}}\Big\}dx + \int_{\mathbb{R}} \left[\frac{\varepsilon\zeta\zeta_x}{\sqrt{\zeta^2+\delta^2}} - \frac{G\zeta^2}{\sqrt{\zeta^2+\delta^2}}\right]\rho_x\,dx.
\end{aligned}
$$

Denote sgn$(\cdot)$ as the signature function

$$\operatorname{sgn}(s) = \begin{cases} 1 & \text{if} \;\; s > 0 \\ 0 & \text{if} \;\; s = 0 \\ -1 & \text{if} \;\; s < 0 \end{cases} .$$

We observe that

$$\lim_{\delta \to 0} \Phi_\delta(s) = \lim_{\delta \to 0} \frac{s}{\sqrt{s^2 + \delta^2}} = \frac{s}{|s|} = \operatorname{sgn}(s).$$

Since $\zeta_x = 0$ almost everywhere (a.e.) on $\{x \mid \zeta(x) = 0\}$, send $\delta \searrow 0$ and use the Lebesgue's dominated convergence theorem we then obtain

$$0 \geqslant \int_{\mathbb{R}} \rho \Big\{ \operatorname{sgn}(\zeta)\zeta - \operatorname{sgn}(\zeta) K * \zeta \Big\} dx + \int_{\mathbb{R}} [\varepsilon \operatorname{sgn}(\zeta) \, \zeta_x - \operatorname{sgn}(\zeta) G(x) \zeta] \rho_x \, dx.$$

Finally, let $\rho_0 \in C_c^\infty(\mathbb{R})$ be a non-negative function satisfying $\rho_0(0) = 1$. Set $\rho(x) = \rho_0(\delta x)$, send $\delta \searrow 0$, then

$$\begin{aligned} &\lim_{\delta \to 0} \rho_0(\delta x) = \rho_0(0) = 1, \\ &\lim_{\delta \to 0} \Big(\rho_0(\delta x) \Big)_x = \lim_{\delta \to 0} \delta \cdot \rho_0'(\delta x) = 0, \end{aligned}$$

and

$$\lim_{x \to \pm\infty} [|\zeta_x| + |\zeta|\,] = 0,$$

we obtain

$$0 \geqslant \int_{\mathbb{R}} \Big\{ \operatorname{sgn}(\zeta)\zeta - \operatorname{sgn}(\zeta) K * \zeta \Big\} dx.$$

As $\zeta(\pm\infty) = 0$, by Lemma 1 we have

$$\int_{\mathbb{R}} \Big[K * |\zeta|(x) - |\zeta|(x) \Big] dx = 0.$$

Note that $|\zeta| = \operatorname{sgn}(\zeta)\zeta$, the above inequality then implies that

$$\int_{\mathbb{R}} \Big[K * |\zeta| - \operatorname{sgn}(\zeta) K * \zeta\Big] dz \leqslant 0.$$

As $|K * \zeta| \leqslant K * |\zeta|$, we derive that

$$K * |\,\zeta| = \operatorname{sgn}(\zeta) K * \zeta \quad \text{in } \ \mathbb{R}.$$

Claim 5. $\zeta \equiv 0$.

Proof. Suppose this is not true. Then $\zeta(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$. Without loss of generality, we assume that $\zeta(x_0) > 0$ and $x_0 < 0$.

As $\zeta(0) = 0$, there exists an open interval $(a, b) \in (-\infty, 0)$ such that $\zeta > 0$ in (a, b), with $\zeta(b) = 0$ and either $a = -\infty$ or $\zeta(a) = 0$. Then

$$K * \zeta = \operatorname{sgn}(\zeta) K * |\zeta| = K * |\zeta| \geqslant 0 \quad \text{ in } (a, b),$$

so that

$$-\varepsilon\zeta_{xx} + G\zeta_x + [1 + G_x]\zeta = K * \zeta \geqslant 0 \quad \text{ in } (a, b).$$

Based on assumption that $\zeta_x(b) \leq 0$, the Hopf Lemma then implies that $\zeta_x(b) < 0$. It follows that for some $\delta > 0$

$$\zeta > 0 \text{ in } (b - \delta, b), \qquad \zeta < 0 \text{ in } (b, b + \delta).$$

Now let $z_0 \in \mathbb{R}$ be a Lebesgue point of K at which $K(z_0) > 0$. Then

$$\begin{aligned}
K * |\zeta|(z_0 + b) + K * \zeta(z_0 + b) &= \int_{\mathbb{R}} K(z_0 - z)[|\zeta(z + b)| + \zeta(z + b)]dz \\
&\geqslant \int_{-\delta}^{\delta} K(z_0 - z)[|\zeta(z + b)| + \zeta(z + b)]dz \\
&= \int_{-\delta}^{0} K(z_0 - z)[|\zeta(z + b)| + \zeta(z + b)]dz \\
&\quad + \int_{0}^{\delta} K(z_0 - z)[|\zeta(z + b)| + \zeta(z + b)]dz \\
&= \int_{0}^{\delta} K(z_0 - z)[2|\zeta(z + b)| + \zeta(z + b)]dz > 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
K * |\zeta|(z_0+b) - K * \zeta(z_0+b) &= \int_{\mathbb{R}} K(z_0-z)[|\zeta(z+b)| - \zeta(z+b)]dz \\
&\geqslant \int_{-\delta}^{\delta} K(z_0-z)[|\zeta(z+b)| - \zeta(z+b)]dz \\
&= \int_{-\delta}^{0} K(z_0-z)[|\zeta(z+b)| - \zeta(z+b)]dz \\
&\quad + \int_{0}^{\delta} K(z_0-z)[|\zeta(z+b)| - \zeta(z+b)]dz \\
&= \int_{-\delta}^{0} K(z_0-z)[2|\zeta(z+b)| - \zeta(z+b)]dz > 0.
\end{aligned}$$

This implies that

$$K * |\zeta|(z_0+b) > |K * \zeta(z_0+b)|$$

and we obtain a contradiction. This contradiction shows that $\zeta \equiv 0$. $\square$

Hence, the solution to (3.1) is unique, which completes the proof of Theorem 7.

4.0 THE LIMIT AS $\varepsilon \searrow 0$

Now we can take $\varepsilon \to 0$ to obtain a solution to (1.5), thereby prove Theorem 1. Recall the problem is

$$\begin{cases} (\, f(\phi) - c\phi\,)_z + \phi = K * \phi \qquad \text{in } \ \mathbb{R}, \\ \phi(-\infty) = u_- \ , \qquad \phi(\infty) = u_+. \end{cases} \tag{1.5}$$

with

$$f \in C^2(\mathbb{R}), \quad K \geqslant 0 \text{ in } \ \mathbb{R}, \quad \int_{\mathbb{R}} K(z)\, dz = 1, \quad \int_{\mathbb{R}} |z| K(z)\, dz < \infty. \tag{1.4}$$

Additionally assume

$$g(s) := f(s) - f(u_-) + \frac{f(u_+) - f(u_-)}{u_+ - u_-}(u_- - s) < 0, \quad \forall\, s \in (u_+, u_-). \tag{1.9}$$

$$c = \frac{f(u_+) - f(u_-)}{u_+ - u_-} + \int_{\mathbb{R}} zK(z)\, dz. \tag{1.8}$$

Theorem 1. *Assume (1.4). For each pair $(u_-, u_+) \in \mathbb{R}^2$ satisfying $u_- > u_+$ and (1.9), there exists a solution (c, ϕ) to (1.5) where c is given by (1.8) and $\phi_x \leqslant 0$ in $\mathbb{R}$ in the distribution sense. In addition, ϕ is an entropy solution in the sense that for every $k \in \mathbb{R}$,*

$$[(F(\phi) - F(k))\mathrm{sgn}(\phi - k)]_x \leq \mathrm{sgn}(\phi - k)(K * \phi - \phi) \tag{1.10}$$

in the distribution sense, where $F(s) := f(s) - cs$.

Moreover, the solution has a jump if

$$\max_{s\in[u_+,u_-]}\Big(-g(s)\Big) > (u_- - u_+)\int_{\mathbb{R}}|z|K(z)\,dz. \qquad (1.11)$$

Proof. We as well divide the proof into two steps in the following sections.

4.1 EXISTENCE

For every $\varepsilon > 0$, let u^ε be the solution to (3.1). The family $\{u^\varepsilon\}_{0<\varepsilon<1}$ is a family of bounded and decreasing functions. By Helly's theorem, there exist a subsequence $\{\varepsilon_j\}$ of positive numbers and a decreasing function ϕ defined in $\mathbb{R}$ such that

$$\begin{aligned}
\lim_{j\to\infty}\varepsilon_j &= 0,\\
\lim_{j\to\infty}u^{\varepsilon_j}(x) &= \phi(x), \quad \forall\, x\in\mathbb{R},\\
\lim_{j\to\infty}u^{\varepsilon_j} &= \phi \qquad \text{in } \; L^2((-M,M)), \quad \forall\, M>0,\\
\lim_{j\to\infty}K*u^{\varepsilon_j}(x) &= K*\phi(x), \quad \forall\, x\in\mathbb{R}.
\end{aligned}$$

To find the limit equation for ϕ, we start from the integral differential equation (3.2). Fix any $a, b\in\mathbb{R}$. By integrating (3.2) over (a, b) we obtain

$$\varepsilon[u^\varepsilon(a) - u^\varepsilon(b)] + \int_a^b \Big\{g(u^\varepsilon(x)) - \int_{\mathbb{R}}K(z)\int_{x-z}^{x}[u^\varepsilon(y) - u^\varepsilon(x)]dydz\Big\}dx = 0.$$

By taking $\varepsilon = \varepsilon_j$ and sending $j\to\infty$ we obtain

$$\int_a^b \Big\{g(\phi(x)) - \int_{\mathbb{R}}K(z)\int_{x-z}^{x}[\phi(y) - \phi(x)]dydz\Big\}dx = 0, \quad \forall a, b\in\mathbb{R}.$$

This implies that for almost every $x \in \mathbb{R}$,

$$g(\phi(x)) - \phi(x) \int_{\mathbb{R}} z K(z)\, dz = - \int_{\mathbb{R}} K(z) \int_{x-z}^{x} \phi(y)\, dy\, dz. \tag{4.1}$$

As the right-hand side is continuous, by redefining ϕ on a countable set (the set of discontinuity of ϕ) we see that the above equation is satisfied for every $x \in \mathbb{R}$. Further as the right-hand side is Lipschitz continuous in x (because it is differentiable with bounded first derivative), by differentiating both side and using the definition of g we obtain

$$\{f(\phi) - c\phi\}_x = K * \phi - \phi \quad \text{in} \;\; L^1_{\mathrm{loc}}(\mathbb{R}).$$

Next, we show that $\phi(\pm\infty) = u_\pm$. Indeed, by sending $x \to \pm\infty$ in (4.1) we obtain

$$g(\phi(\pm\infty)) = 0.$$

As

$$u_+ \leqslant \phi(\infty) \leqslant \tfrac{1}{2}(u_+ + u_-) \leqslant \phi(-\infty) \leqslant u_-,$$

and

$$g(s) < 0, \qquad \forall s \in (u_+, u_-),$$

we must have

$$\phi(\pm\infty) = u_\pm.$$

Thus, ϕ is a solution to (1.5). In addition, the solution satisfies $\phi_x \leqslant 0$ in the distribution sense, which is a consequence from the previous chapter.

4.2 ENTROPY SOLUTION

Now we show that ϕ is an entropy solution, i.e., it satisfies (1.10).

For this, let $k \in \mathbb{R}$ and $\Phi \in C^2(\mathbb{R})$ be any convex function, i.e., $\Phi''(s) \geqslant 0$ for all $s \in \mathbb{R}$. Denote

$$\eta(s) = \int_k^s \Phi'(t)\Big[f'(t) - c\Big]dt, \qquad \forall\, s \in \mathbb{R}.$$

Then by using the equation for u^ε we have

$$\begin{aligned} 0 &= \Phi'(u^\varepsilon)\Big\{ - \varepsilon u^\varepsilon_{xx} + [f'(u^\varepsilon) - c]u^\varepsilon_x + u^\varepsilon - K * u^\varepsilon \Big\} \\ &= -\Big[\varepsilon\Phi(u^\varepsilon)\Big]_{xx} + \varepsilon\Phi''(u^\varepsilon)\Big(u^\varepsilon_x\Big)^2 + \Big[\eta(u^\varepsilon)\Big]_x + \Phi'(u^\varepsilon)\Big[u^\varepsilon - K * u^\varepsilon\Big]. \end{aligned}$$

Hence, for any non-negative smooth function $\rho \in C^\infty(\mathbb{R})$ with compact support, after dropping the non-negative term $\rho\varepsilon\Phi''(u^\varepsilon)\Big(u^\varepsilon_x\Big)^2$ we have

$$\begin{aligned} 0 \geqslant &\int_{\mathbb{R}} \rho\Big\{ - \Big[\varepsilon\Phi(u^\varepsilon)\Big]_{xx} + \Big[\eta(u^\varepsilon)\Big]_x + \Phi'(u^\varepsilon)\Big[u^\varepsilon - K * u^\varepsilon\Big]\Big\}\, dx \\ = &\int_{\mathbb{R}} \Big\{\rho_x(\varepsilon\Phi(u^\varepsilon))_x - \eta(u^\varepsilon)\rho_x + \rho\Phi'(u^\varepsilon)\Big[u^\varepsilon - K * u^\varepsilon\Big]\Big\}\, dx \\ = &\int_{\mathbb{R}} \Big\{ - \rho_{xx}\varepsilon\Phi(u^\varepsilon) - \eta(u^\varepsilon)\rho_x + \rho\Phi'(u^\varepsilon)\Big[u^\varepsilon - K * u^\varepsilon\Big]\Big\}\, dx. \end{aligned}$$

Set $\varepsilon = \varepsilon_j$ and send $j \to \infty$ we then obtain

$$\int_{\mathbb{R}} \Big\{ - \eta(\phi)\rho_x + \Phi'(\phi)[\phi - K * \phi]\rho\Big\}dx \leqslant 0.$$

Take the particular choice

$$\Phi(s) = \Phi_\delta(s) := \sqrt{(s-k)^2 + \delta^2}.$$

Then

$$\Phi'_\delta(s) = \frac{s-k}{\sqrt{(s-k)^2+\delta^2}}.$$

Hence by sending $\delta \searrow 0$, we derive by the Lebesgue's dominated convergence theorem that

$$\int_{\mathbb{R}} \Big\{ -[F(\phi)-F(k)]\mathrm{sgn}(\phi-k)\rho_x + \mathrm{sgn}(\phi-k)[\phi - K*\phi]\rho \Big\} dx \leqslant 0$$

since

$$\begin{aligned}
\lim_{\delta\to 0}\Phi_\delta(s) &= |s-k|, \qquad \lim_{\delta\searrow 0}\Phi'_\delta(s) = \mathrm{sgn}(s-k),\\
\lim_{\delta\to 0}\eta_\delta(s) &= \lim_{\delta\to 0}\int_k^s \Phi'_\delta(t)[f'(t)-c]dt = \mathrm{sgn}(s-k)[F(s)-F(k)].
\end{aligned}$$

Thus, we have

$$\int_{\mathbb{R}} \Big\{ \big[(F(\phi)-F(k))\mathrm{sgn}(\phi-k)\big]_x + \mathrm{sgn}(\phi-k)\big[\phi - K*\phi\big] \Big\}\rho\, dx \leqslant 0.$$

It follows directly that

$$[\mathrm{sgn}(\phi-k)(F(\phi)-F(k))]_x \leqslant \mathrm{sgn}(\phi-k)[K*\phi-\phi]$$

in the distribution sense.

Finally, we see from (4.1) that

$$-g(\phi(x)) = \int_{\mathbb{R}} K(z)\int_{x-z}^{x}[\phi(y)-\phi(x)]dy\, dz < [u_- - u_+]\int_{\mathbb{R}} |z|K(z)\, dz.$$

Hence, if (1.11) holds, then

$$
\begin{aligned}
-g(\phi(x)) &< [u_- - u_+]\int_{\mathbb{R}} |z|K(z)\,dz \\
&< \max_{s\in\mathbb{R}} \frac{(s-u_+)f(u_-) + (u_- - s)f(u_+) - (u_- - u_+)f(s)}{u_- - u_+} \\
&= \max_{s\in\mathbb{R}} \Big(-g(s)\Big),
\end{aligned}
$$

i.e.,

$$
|g(\phi(x))| > \max_{s\in[u_+,u_-]} |g(s)|, \quad \forall\, x \in \mathbb{R}.
$$

This implies that ϕ cannot be continuous in $\mathbb{R}$, and so far we have completed the proof of Theorem 1.

4.3 UNIQUENESS

Here we show that entropy solutions are unique, using Serre's approach [31].

Let ϕ be the decreasing entropy solution obtained as above. Let ψ be any entropy solution. Then following a standard yet highly technical computation, one finds that

$$
[\mathrm{sgn}(\phi-\psi)(F(\phi)-F(\psi)]_x \leqslant \mathrm{sgn}(\phi-\psi)[K*(\phi-\psi)-(\phi-\psi)]
$$

in the distribution sense. Hence, by setting $\zeta=\phi-\psi$, integrating the inequality over $(-L,L)$ and sending $L\to\infty$ we derive that

$$
\lim_{L\to\infty}\int_{-L}^{L} \mathrm{sgn}(\zeta)\Big\{K*\zeta-\zeta\Big\}dx \geqslant 0.
$$

Since $\lim_{x\to\pm\infty} |\zeta(x)| = 0$, we have

$$\lim_{L\to\infty} \int_{-L}^{L} [K * |\zeta| - |\zeta|] dx = 0.$$

Thus

$$\int_{\mathbb{R}} \Big\{ \mathrm{sgn}(\zeta) K * \zeta - K * |\zeta| \Big) dx \geqslant 0.$$

As $K * |\zeta| \geqslant |K * \zeta|$, we then derive that

$$K * |\zeta| = \mathrm{sgn}(\zeta) K * \zeta.$$

This implies that either $\zeta \geqslant 0$ in $\mathbb{R}$ or $\zeta \leqslant 0$ in $\mathbb{R}$. Indeed, if there are Lebesgue points x_1 and x_2 of ζ such that $\zeta(x_1)\zeta(x_2) < 0$, then by working on translation of $\zeta^h(x) = \phi(x-h) - \psi(x)$ one can find an appropriate h and points x_0 such that $\zeta^h(\cdot)$ changes signs near x_0, so that if z_0 is a Lebesgue point of K, then

$$K * |\zeta^h|(x_0 - z_0) > |K * \zeta^h(x_0 - z_0)|.$$

But this would contradicts the conclusion that $K * |\zeta^h| = \mathrm{sgn}(\zeta^h) K * \zeta^h$. Hence for every $h \in \mathbb{R}$, either

$$\phi(\cdot - h) - \psi(\cdot) \geqslant 0 \quad \text{on a.e. in } \mathbb{R}$$

or

$$\phi(\cdot - h) - \psi(\cdot) \leqslant 0 \quad \text{on a.e. in } \mathbb{R}.$$

Consequently, there exists $h_0 \in \mathbb{R}$ such that

$$\phi(\cdot - h_0) = \psi(\cdot) \quad \text{a.e. in } \mathbb{R}.$$

Thus, entropy solution is unique up to a translation.

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